

ON INTEGER SEQUENCES IN PRODUCT SETS

SAI TEJA SOMU

ABSTRACT. Let B be a finite set of natural numbers or complex numbers. Product set corresponding to B is defined by $B.B := \{ab : a, b \in B\}$. In this paper we give an upper bound for longest length of consecutive terms of a polynomial sequence present in a product set accurate up to a positive constant. We give a sharp bound on the maximum number of Fibonacci numbers present in a product set when B is a set of natural numbers and a bound which is accurate up to a positive constant when B is a set of complex numbers.

1. INTRODUCTION

In [3] and [4] the author has proved that if B is a set of natural numbers then the product set corresponding to B cannot contain long arithmetic progressions. In [3] it was shown that the longest length of arithmetic progression is at most $O(|B| \log |B|)$. We try to generalize this result for polynomial sequences. Let $P(x) \in \mathbb{Z}[x]$ be a polynomial with positive leading coefficient. Let R be the longest length of consecutive terms of the sequence, that is,

$$R = \max\{n : \text{there exists an } x \in \mathbb{N} \text{ such that } \{P(x+1), \dots, P(x+n)\} \subset B.B\}.$$

We prove that R cannot be large for every polynomial $P(x)$. In section 2 we consider the question of determining maximum number of Fibonacci and Lucas sequence terms in a product set.

As in [3] we define an auxiliary bipartite graph $G(A, B.B)$ and auxiliary graph $G'(A, B.B)$ which are constructed for any sets A and B whenever $A \subset B.B$. The color classes of G are two copies of B whereas G' has only one color copy of B and for each $a \in A$ we pick a unique representation $a = b_1 b_2$ and place an edge (b_1, b_2) in G and in G' . Note that $V(G) = 2|B|$, $V(G') = |B|$ and $E(G) = E(G') = |A|$. Observe that G' can have self loops and G cannot have self loops.

2. NUMBER OF FIBONACCI NUMBERS AND LUCAS NUMBERS

Let B be a finite set of natural numbers. Let A be the set of Fibonacci numbers contained in the product set. From [2] there are only two perfect square Fibonacci numbers, viz., 1 and 144. Hence there can be at most two self loops in the graph $G'(A, B.B)$. We give an upper bound on cardinality of A by using the following lemma.

Lemma 2.1. *Let F_n and F_m be n th and m th Fibonacci numbers and $m < n$ and $n > 2$ then $\gcd(F_n, F_m) < \sqrt{F_n}$.*

Proof. Let $d = \gcd(m, n)$. From the strong divisibility property of Fibonacci numbers $\gcd(F_n, F_m) = F_d$. We know that $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Since $m < n$, clearly $d \leq \frac{n}{2}$. If $d = 1$ then the hypothesis is clearly true and if $d > 1$, we have

$$(F_d)^2 = \frac{(\alpha^d - \beta^d)^2}{(\alpha - \beta)^2} < \frac{(\alpha^{2d} - \beta^{2d})}{(\alpha - \beta)} \leq F_n.$$

Thus, $\gcd(F_m, F_n) < \sqrt{F_n}$. \square

Theorem 2.2. *There cannot be more than $|B|$ Fibonacci numbers in the product set $B.B$ when B is a set of natural numbers.*

Proof. We claim that in the graph $G'(A, B.B)$ there cannot any cycle other than self loops. Suppose there is a k -cycle $b_1 b_2 \cdots b_k b_1$ which implies that $b_i b_{i+1}$ for $1 \leq i \leq k-1$ and $b_k b_1$ are distinct Fibonacci numbers in the set $B.B$. Without loss of generality let us assume $b_1 b_2 = F$ is the largest Fibonacci number among $b_i b_{i+1}$ for $1 \leq i \leq k-1$ and $b_k b_1$. From Lemma 2.1, we have

$$\begin{aligned} b_1 &\leq \gcd(b_1 b_2, b_1 b_k) < \sqrt{F}, \\ b_2 &\leq \gcd(b_1 b_2, b_2 b_3) < \sqrt{F}. \end{aligned}$$

Hence $F = b_1 b_2 < F$ which is a contradiction. Hence there cannot be any cycle. From [2] there cannot be more than 2 self loops. Hence the number of edges which equal number of Fibonacci numbers in the set $B.B$ cannot exceed $|B| + 1$.

Now we prove that there cannot be $|B| + 1$ Fibonacci numbers. Suppose there are $|B| + 1$ Fibonacci numbers, as the graph cannot have any cycle there should be two self loops namely, 1 and 12 and the graph obtained by removing the two self loops should be connected tree of $|B|$ vertices. Since the graph is connected there should be a path between 1 and 12. Let the path be $b_1 b_2 \cdots b_k$ which implies that $b_i b_{i+1}$ for $1 \leq i \leq k-1$ are Fibonacci numbers and without loss of generality assume $b_1 = 1$ and $b_k = 12$. Let l be the index of highest value of $b_i b_{i+1}$. Clearly $l \neq 1$ and if $2 \leq l \leq k-2$ then from Lemma 2.1

$$\begin{aligned} b_l &\leq \gcd(b_l b_{l+1}, b_{l-1} b_l) < \sqrt{b_l b_{l+1}}, \\ b_{l+1} &\leq \gcd(b_l b_{l+1}, b_{l+1} b_{l+2}) < \sqrt{b_l b_{l+1}}. \end{aligned}$$

Which implies $b_l b_{l+1} < b_l b_{l+1}$. Hence $l = k-1$. Again from Lemma 2.1

$$b_{k-1} \leq \gcd(b_{k-1} b_k, b_{k-2} b_{k-1}) < \sqrt{b_{k-1} b_k}$$

which implies $b_{k-1} < b_k = 12$ but there are no Fibonacci numbers of the form $12b$ with $b < 12$. Hence there cannot be $|B| + 1$ Fibonacci numbers. Thus number of Fibonacci numbers in the set $B.B$ is $\leq |B|$. \square

Now we consider the case where B is a set of complex numbers and try to give an upper bound on the number of Lucas sequence terms in the product set. Let A be the set of Lucas sequence terms with indices greater than 30 in the product set $B.B$.

Lemma 2.3. *There cannot be any cycle in $G(A, B.B)$.*

Proof. Suppose there was a cycle $b_1 b_2 \cdots b_{2k} b_1$. Then

$$\begin{aligned} b_1 b_2 &= L_{n_1} \\ b_2 b_3 &= L_{n_2} \\ &\vdots \\ &\vdots \\ b_{2k} b_1 &= L_{n_{2k}}, \end{aligned}$$

where L_{n_i} are Lucas sequence terms with indices greater than 30, which implies

$$(1) \quad \prod_{i=1}^k L_{n_{2i}} = \prod_{j=1}^k L_{n_{2j-1}}$$

Let n_i be the largest index ≥ 31 . Then from [1] L_{n_i} contains a primitive divisor p and hence p divides exactly one side of (1) and therefore (1) cannot be true. Thus there cannot be any cycle. \square

Theorem 2.4. *Let $(L_n)_{n=1}^\infty$ be a Lucas sequence. Then number of distinct elements of $(L_n)_{n=1}^\infty$ in $B.B$ is less than $2|B| + 30$.*

Proof. Since number of vertices in $G(A, B.B)$ is $2|B|$ and from Lemma 2.3 there cannot be a cycle in $G(A, B.B)$ the number of edges in $G(A, B.B) \leq 2|B| - 1$. Hence the number of distinct terms in Lucas sequence of index ≥ 31 is $\leq 2|B| - 1$. Hence number of distinct Lucas sequence terms in $B.B$ is $\leq 2|B| + 29$. \square

3. POLYNOMIAL SEQUENCES

Now we turn onto the second problem in this paper. Given a polynomial $P(x)$ with positive leading coefficient and integer coefficients what can we say about the longest length of consecutive terms in the product set $B.B$.

Since there can be at most finitely r such that $P(r) \leq 0$ or $P'(r) \leq 0$ there exists an l such that $P(r+l) > 0$ and $P'(r+l) > 0$ for all $r \geq 1$. Hence we can assume without loss of generality that every irreducible factor $f(x)$ of $P(x)$ $f(x) > 0$ and $f'(x) > 0 \ \forall x \geq 1$ as this assumption only effects R by a constant. From now we will be assuming that for every irreducible divisor $f(x)$ of $P(x)$ $f(x) > 0$ and $f'(x) > 0$ for all natural numbers x . We prove three lemmas in order to give an upper bound for R .

If $f(x) \in \mathbb{Z}[x]$ is an irreducible polynomial of degree ≥ 2 . Let D be the discriminant of $f(x)$. Let d be the greatest common divisor of the set $\{f(n) : n \in \mathbb{N}\}$. Let $f_1(x) = \frac{f(x)}{d}$. Denote Dd^2 by M . If $p^e \parallel M$ then $p^e \nmid d$ and hence there exists an a_p , such that $f_1(x)$ is not divisible by p for all $x \equiv a_p \pmod{p^e}$. From Chinese remainder theorem there exists an integer a such that $a \equiv a_p \pmod{p^e}$ for all primes dividing M and hence there exists an a such that $f_1(x)$ is relatively prime to M for all $x \equiv a \pmod{M}$.

Lemma 3.1. *For sufficiently large R the number of numbers in the set $\{f_1(r+i) : 1 \leq i \leq R, r+i \equiv a \pmod{M}\}$ with atleast one prime factor greater than R is $\geq \frac{R}{3M}$ for every non negative integer r .*

Proof. Let

$$Q = \prod_{\substack{i=1 \\ r+i \equiv a \pmod{M}}}^R f_1(r+i).$$

Let S be the largest divisor of Q with all prime factors $\leq R$. Let e_p be the index of p in S . Let $\rho(p)$ denote the number of solutions modulo p of the congruence $f(x) \equiv 0 \pmod{p}$.

$$\begin{aligned} \log S &= \sum_{\substack{p \nmid M \\ p \leq R}} e_p \log p \\ &= \sum_{\substack{p \nmid M \\ p \leq R}} \sum_{n=1}^{\lfloor \frac{\log f(r+R)}{\log p} \rfloor} \sum_{\substack{i=1 \\ r+i \equiv a \pmod{M} \\ f_1(r+i) \equiv 0 \pmod{p^n}}}^R \log p \\ &= \sum_{\substack{p \nmid M \\ p \leq R}} \sum_{n=1}^{O(\frac{\log(r+R)}{\log p})} \left(\frac{\rho(p) \log p R}{M p^n} + O(\log p) \right) \\ &= \sum_{\substack{p \nmid M \\ p \leq R}} \frac{\rho(p) \log p R}{M p} + O\left(\frac{\log(r+R)R}{\log R} \right). \end{aligned}$$

From prime ideal theorem, we have

$$\sum_{\substack{p \nmid M \\ p \leq R}} \frac{\rho(p) \log p R}{M p} = \frac{R \log R}{M} + O(R).$$

Thus, we have

$$\log S = \frac{R \log R}{M} + O\left(\frac{\log(r+R)R}{\log R} \right).$$

Let L be a subset of $\{f_1(r+i) : 1 \leq i \leq R, r+i \equiv a \pmod{M}\}$ containing all the numbers which do not contain any prime factor greater than R and let l denote the cardinality of L .

$$\begin{aligned} \log \prod_{\substack{i=1 \\ f_1(r+i) \in L}}^R f_1(r+i) &\geq \log \prod_{i=1}^l f_1(r+i) \\ &= n \sum_{i=1}^l \log(r+i) + O(l) \\ &= nl \log(r+l) + O(l), \end{aligned}$$

where n is the degree of the polynomial $f(x)$. Hence

$$nl \log(r+l) + O(l) \leq \frac{R \log R}{M} + O\left(\frac{\log(r+R)R}{\log R} \right).$$

Hence for sufficiently large R , l should be less than $\frac{2R}{3M} - 2$. Hence number of numbers belonging to the set $\{f_1(r+i) : 1 \leq i \leq R, r+i \equiv a \pmod{M}\}$ with atleast one prime factor greater than R is $\geq \frac{R}{3M}$. \square

The following corollary immediately follows from Lemma 3.1.

Corollary 3.2. *If $P(x)$ has an irreducible divisor of degree ≥ 2 . Then there exist $\Omega(R)$ numbers in the set $\{P(r+i) : 1 \leq i \leq R\}$ with atleast one prime factor greater than R .*

Lemma 3.3. *If $f(x)$ is a linear polynomial. If $r \geq R^\gamma$ for a $\gamma > 1$ then there exists a constant $c > 0$ depending upon γ such that for sufficiently large R , number of numbers of the set $\{f(r+i) : 1 \leq i \leq R\}$ with a prime factor greater than R is greater than cR .*

Proof. The proof is similar to that of Lemma 3.1. Let $Q = \prod_{i=1}^R f(r+i)$ and S be the largest divisor of Q with all prime factors $\leq R$.

$$\begin{aligned} \log S &= \sum_{p \leq R} e_p \log p \\ &= \sum_{p \leq R} \sum_{n=1}^{O\left(\frac{\log(r+R)}{\log p}\right)} \sum_{\substack{i=1 \\ f(r+i) \equiv 0 \pmod{p^n}}}^R \log p \\ &= \sum_{p \leq R} \frac{R \log p}{p} + O\left(\frac{R \log(r+R)}{\log R}\right) \\ &= R \log R + O\left(\frac{R \log(r+R)}{\log R}\right). \end{aligned}$$

Let L be a subset of $\{1 \leq i \leq R\}$ containing all i such that $f(r+i)$ has all prime factors $\leq R$. Let the cardinality of L be l .

$$\begin{aligned} \log \prod_{\substack{i=1 \\ i \in L}}^R f(r+i) &\geq \log \prod_{i=1}^l f(r+i) \\ &= l \log(r+R) + O(R). \end{aligned}$$

which implies

$$l \log(r+R) + O(R) \leq R \log R + O\left(\frac{R \log(r+R)}{\log R}\right).$$

For sufficiently large R , l should be $\leq \frac{(1+\gamma)}{2\gamma} R$. Hence for sufficiently large R number of numbers of the set $\{f(r+i) : 1 \leq i \leq R\}$ with atleast one prime factor greater than R is $\geq \frac{(\gamma-1)R}{2\gamma}$. \square

We have the following Corollary for Lemma 3.3.

Corollary 3.4. *If degree of every irreducible divisor of $P(x)$ is 1 and $r \geq R^\gamma$ then number of elements of the set $\{P(r+i) : 1 \leq i \leq R\}$ with atleast one prime factor greater than R is $\Omega(R)$.*

Lemma 3.5. *Let $f(x)$ be a linear polynomial. If $r \leq R^\gamma$ for some $\gamma > 1$ then there are $\Omega\left(\frac{R}{\log R}\right)$ numbers of the set $\{f(r+i) : 1 \leq i \leq R\}$ with atleast one prime factor greater than $\frac{R}{2}$.*

Proof. Let $f(n) = an + b$ then there are $\Omega(\frac{R}{\log R})$ primes between $(\frac{R}{2}, R]$ which are coprime to a . Each prime has one or two $i \in [1, R]$ such that $p|f(r+i)$. For each $f(r+i)$ there are at most $O(1)$ prime divisors belonging to $(\frac{R}{2}, R]$. Hence there are $\Omega(\frac{R}{\log R})$ numbers with atleast one prime factor greater than $\frac{R}{2}$. \square

Corollary 3.6. *If degree of every irreducible divisor of $P(x)$ is 1 and $r \leq R^\gamma$ then number of elements of the set $\{P(r+i) : 1 \leq i \leq R\}$ with atleast one prime factor belonging to the range $(\frac{R}{2}, R]$ is $\Omega(\frac{R}{\log R})$.*

In a graph $G(V, E)$ for $v \in V$ we define $V(v)$ to be the set of all vertices adjacent to v .

Lemma 3.7. *If there is a bipartite graph (A, B, E) such that for all $a \in A$ and $b \in B$, degree of a is $\leq n$ and degree of b is ≥ 1 then there exists a sequence of vertices b_1, \dots, b_k with $b_i \in B$ satisfying $V(b_1) \neq \phi$ and $V(b_i)/(\cup_{j=1}^{i-1} V(b_j)) \neq \phi$ for $2 \leq i \leq k$ and $k \geq \frac{|B|}{n}$.*

Proof. The proof is by induction on n . For $n = 1$ the lemma is true since degree of $a \leq 1 \quad \forall a \in A \implies V(b_1) \cap V(b_2) = \phi \quad \forall b_1 \neq b_2 \in B$ and the sequence $b_1, \dots, b_{|B|}$ will clearly satisfy $V(b_1) \neq \phi$ and $V(b_i)/(\cup_{j=1}^{i-1} V(b_j)) \neq \phi$ for $2 \leq i \leq k$. If the lemma is true for $n = r$ we have to prove for $n = r + 1$. Order the vertices of B as $b_1, \dots, b_{|B|}$. Let $S = \{a \in A : \text{degree of } a \geq 1\}$. Let $S_1 = V(b_1)$ and for $2 \leq i \leq |B|$, let $S_i = V(b_i)/(\cup_{j=1}^{i-1} V(b_j))$. Observe that $S = \cup_{i=1}^{|B|} S_i$. Let K be a set defined by $K = \{b_i : S_i \neq \phi\}$. If $|K| \geq \frac{|B|}{r+1}$ then we can choose the vertices in the set K arranged in a sequence which satisfies the hypothesis. If $|K| < \frac{|B|}{r+1}$ then consider the induced subgraph $A \cup (B - K)$ then degree of a is less than or equal to r for all $a \in A$. From the induction assumption there exists a sequence with length $\geq \frac{|B-K|}{r} > |B|(1 - \frac{1}{r+1})\frac{1}{r} = \frac{|B|}{r+1}$ in $B - K$ satisfying the hypothesis which completes the proof by induction. \square

Theorem 3.8. *Let $P \in \mathbb{Z}[x]$ and has a positive leading coefficient and if $\{P(r+1), \dots, P(r+R)\}$ is contained in the product set $B \cdot B$ for a nonnegative integer r and natural number R and B is a set of complex numbers.*

(1) *If P has an irreducible factor of degree ≥ 2 then $R = O(|B|)$.*

(2) *If P has no irreducible factor of degree ≥ 2 and $r > R^\gamma$ and $\gamma > 1$ then $R = O(|B|)$.*

(3) *If P has no irreducible factor of degree ≥ 2 and $r \leq R^\gamma$ and $\gamma > 1$ then $R = O(|B| \log |B|)$.*

Proof. If P has an irreducible factor f of degree greater than 2 or $P(x)$ has no irreducible divisor of degree ≥ 2 and $r > R^\gamma$ let

$$A = \{p : p \text{ is a prime, } p|P(r+i) \text{ for some } 1 \leq i \leq R, p > R\}$$

and let

$$C = \{P(r+i) : 1 \leq i \leq R, \exists \text{ prime } p > R \text{ such that } p|P(r+i)\}.$$

If $P(x)$ has no irreducible divisor of degree ≥ 2 and $r \leq R^\gamma$ then let

$$A = \{p : p \text{ is a prime, } \frac{R}{2} < p \leq R \text{ and } p|P(r+i) \text{ for some } 1 \leq i \leq R\}$$

and let

$$C = \{P(r+i) : 1 \leq i \leq R, \exists \text{ prime } p \in (\frac{R}{2}, R] \text{ such that } p|P(r+i)\}.$$

In cases (1) and (2) from Corollaries 3.2 and 3.4 the size of C is $\Omega(R)$. In case (3) from Corollary 3.6 the size of C is $\Omega(\frac{R}{\log R})$. If we consider a bipartite graph G between $A \cup C$ constructed such that there exists an edge $p \in A$ and $P(r+i) \in C$ if and only if $p|P(r+i)$. In this graph the degree of $a \in A$ is less than or equal to the degree of polynomial P . Hence from Lemma 3.7 there exists a sequence c_1, c_2, \dots, c_k with $k \geq \frac{|C|}{\text{degree of } P}$ such that $V(c_1) \neq \phi$ and $V(c_i) \cap \cup_{j=1}^{i-1} V(c_j) = \phi$. Therefore every c_i has a prime divisor which does not divide any of c_j for $1 \leq j \leq i-1$. Let $C' = \{c_1, \dots, c_k\}$. Note that in cases (1) and (2) $|C'| = \Omega(R)$ and in case 3 $|C'| = \Omega(\frac{R}{\log R})$.

Consider the bipartite auxiliary graph $G(C', B.B)$. We claim that there cannot be any cycle in this graph. Suppose there was a cycle $b_1 b_2 \dots b_{2k} b_1$ then

$$b_1 b_2 = c_{n_1}$$

$$b_2 b_3 = c_{n_2}$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$b_{2k} b_1 = c_{n_{2k}}$$

and

$$(2) \quad \prod_{i=1}^k c_{n_{2i}} = \prod_{j=1}^k c_{n_{2j-1}}$$

let n_i be the highest index present in the cycle. There exists a prime p such that $p|c_{n_i}$ and $p \nmid c_{n_j}$ for $j \neq i$ and hence p divides exactly one side of (2) and hence (2) cannot be true. Thus there exists no cycle in $G(C', B.B)$. Hence $|C'| \leq 2|B| - 1$. Therefore $R = O(|B|)$ in cases (1),(2) and $R = O(|B| \log |B|)$ in case (3) which completes the proof of the theorem. \square

REFERENCES

1. Y. Bilu, G. Hanrot and P. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers*. J. Reine Angew. Math. **539**(2001), 75-122.
2. Y. Bugeaud, M. Mignotte, and S. Siksek, *Classical and modular approaches to exponential Diophantine equations. I. Fibonacci and Lucas perfect powers*, Ann. of Math. (2) **163**:3 (2006), 969-1018.
3. D. Zhelezov, *Improved bounds for arithmetic progressions in product sets*, to appear in Int.J. Number Theory.
4. D. Zhelezov, *Product sets cannot contain long arithmetic progressions*, Acta Arith. 163 (2014), 299-307.

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY ROORKEE, INDIA 247667